Central Extensions

A abelian group

G group

Def

An extension E of G by A is a s.e.s of groups

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

central if $\mathrm{im}(i)$ is in center of G.

E.g.

1. $A \times G$ trivial extension
2. $A \rtimes G$

Poincaré group $SO(1,3) \ltimes \mathbb{R}^4$.

Not central since e.g rotating and translating don't commute.

3. $1 \to \{\pm 1\} \to SL(2,\mathbb{C}) \xrightarrow{\pi} SO(1,3) \to 1$ is central.

Special case of FUN fact:

G connected Lie group

E universal covering of G

Then E is extension of G by group of deck transformations which is iso to $\pi_1(G)$.

Quantization of symmetries

"Quantum theory is projective,
Quantization is linear"

Dan Freed 2307.08147.

Sketchy sketch of situation:

Given vector space W ($\in \mathrm{Vect}_{\mathbb{C}}^{f.d.}$ for simplicity)

A linear symmetry of W induces projective symmetry of $\mathbb{P}(W)$.

projective symmetry of $\mathbb{P}(W)$ has a $\mathbb{C}^\times$-torsor of lifts to linear symmetry.

i.e. we have s.e.s of Lie groups

$$0 \to \mathbb{C}^\times \to GL \to PGL \to 0$$

$$PGL \cong GL/Z \qquad Z = \mathbb{C}^\times \mathrm{id}.$$

Linearization of projective action is a lift

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r] \arrow[d, equal] & GL \arrow[r] & PGL \\
\mathbb{C}^\times \arrow[r] & \tilde{G} \arrow[u, dashed] \arrow[r] & G \arrow[u] \arrow[ul] \arrow[l, dashed, bend right, "L"']
\end{tikzcd}$$

which is equivalently a splitting of the lower central extension.

Hence the obstruction to linearization is a hom $G \longrightarrow B\mathbb{C}^\times$ which is measured by $H^2(G;\mathbb{C}^\times)$.

In Classical to Quantum roughly

Have a classical system with phase space Y and symmetry G.

i.e $\tau: G \longrightarrow Aut(Y)$ (usually Lie groups)

Cannonical quantization means to find a Hilbert space $\mathcal{H}$ such that the functions on Y (classical observables) act as (self-adjoint) operators on $\mathcal{H}$ with {Poisson Bracket} = [Commutator]

After quantization it is an assumption that we get a symmetry

$T: G \longrightarrow U(\mathbb{P}(\mathcal{H}))$

which is (cts, strong op topology on $U(\mathbb{P}(\mathcal{H}))$

and preserves the transition amplitude

Wigner-von Neumann Theorem

$$U(1) \longrightarrow U(\mathcal{H}) \longrightarrow U(\mathbb{P})$$

fits into central extension

and every $G \longrightarrow U(\mathbb{P})$

lifts to $G \longrightarrow U(\mathcal{H})$.

Hence in order to classify all projective representations we can instead classify linear representations $\tilde{G} \longrightarrow U(\mathcal{H})$ by story similar to previous argument.

The allowable "quantum symmetries" are then induced by representations of central extensions.

Similarly for Lie algebras
we have central extensions of $\mathfrak{g}$ by
an abelian Lie algebra.
and central extension of Lie groups
induces central extension of Lie algebras.
In particular

$$1 \to U(1) \to \tilde{G} \to G \to 1$$

$$\Rightarrow \quad 0 \to \mathbb{R} \to \mathrm{Lie}(\tilde{G}) \to \mathrm{Lie}(G) \to 0$$

and in good cases $H^2(G,\mathbb{R}) \cong H^2(\mathrm{Lie}(G),\mathbb{R})$

Virasoro & Witt

Recall last time

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$$

Given classical theory with $\mathrm{Conf}(\mathbb{R}^{1,1})$ we want to study $\mathrm{Diff}_+(S^1)$ and $\mathrm{Lie}(\mathrm{Diff}(S^1))$ and central extensions of them.

$\mathrm{Diff}_+(S^1)$ is infinite dim Lie Group

of real vector fields on S^1, $\mathrm{Vect}(S^1)$

In fact $\mathrm{Vect}(S^1)$ is already a lie algebra and isomorphic (up to sign) to $\mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

$\mathrm{Vect}(S^1)$ is spanned by all $A(\theta)\partial_\theta$.

Using fourier series we can expand $A(\theta)$ in terms of $\{e^{in\theta}\}_{n\in\mathbb{Z}}$.

hence define $L_n := -ie^{in\theta}\partial_\theta \in \mathrm{Vect}^{\mathbb{C}}(S^1)$

$W := \mathrm{span}_{\mathbb{C}}\{L_n \mid n\in\mathbb{Z}\}$ is called the Witt algebra.

$$[L_n, L_m] = e^{i(n+m)\theta}\, i(n-m)\partial_\theta$$
$$= (m-n)\, L_{n+m}$$

Hence the Witt algebra W is part of complexified Lie algebra

$$\mathrm{Vect}^{\mathbb{C}}(S^1) \times \mathrm{Vect}^{\mathbb{C}}(S^1)$$

belonging to the classical conformal symmetry.

<u>Remark</u>

Since conformal transformations are hol/antihol maps with non vanishing derivative. These infinitesimally can be written as $z \to z + \sum_{n \in \mathbb{Z}} a_n z^n$

Laurent gives rise to Lie algebra with basis $L_n := z^{1-n}$

and this can give you that W is dense subalgebra of Lie algebra of hol vector fields on $\mathbb{C} \setminus 0$.

Virasoro

By previous considerations we look at proper central extensions of W by $\mathbb{C}$.

Theorem (Gelfand Fuks 68')

$H^2(W, \mathbb{C}) \cong \mathbb{C}$

spanned by $\omega : W \times W \longrightarrow \mathbb{C}$

$$(L_n, L_m) \mapsto \delta_{n+m} \frac{n}{12}(n^2-1)$$

Proof

Calculation.

Remark

The $\frac{1}{12}$ factor is convention coming from ζ function regularization.

__Def__

Vir the Virasoro algebra is the central extension of W by ω

$$\mathrm{Vir} = W \oplus \mathbb{C} Z$$

$$[L_n, L_m] = (n-m) L_{n+m} + \delta_{n+m} \frac{n}{12}(n^2-1) Z$$

$$[L_n, Z] = 0 \text{ by definition.}$$

__Remark__

Vir is not the Lie algebra of any complex Lie Group.

Firstly because it is infinite dimensional and we could choose lots of different topologies compatible with the structure.

+ other technicalities in the book.

BUT

there does exist a real Lie Group $\mathcal{F}$ s.t. $\mathrm{Lie}(\mathcal{F}) = \mathrm{Vir}^{\mathbb{R}}$

and $\mathcal{F}$ is central extension of $\mathrm{Diff}_+(S^1)$ by S^1.